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Smoothest Interpolation in the Mean

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We consider the problem of characterizing those functions that sequentially take on given mean values and whose rth derivatives have minimal L_p-norm for $p \in (1, \infty]$. © 1999 Academic Press

1. INTRODUCTION

Let $r \geqslant 2$ be fixed. For $p \in (1, \infty]$, $W_p^r[0, 1]$ denotes the usual Sobolev space

$$\{f \in C^{r-1}[0, 1]: f^{(r-1)} \text{ is absolutely continuous and } \|f^{(r)}\|_p < \infty\}.$$

Let $f_1, ..., f_n$ be given fixed real data, $f_i \neq f_{i+1}$, $i = 1, ..., n-1$. Set

$$\Xi_n = \{\mathbf{t} = (t_1, ..., t_n): 0 \leqslant t_1 < \cdots < t_n \leqslant 1\}.$$

For each $\mathbf{t} \in \Xi_n$, set

$$F_p^r(\mathbf{t}; \mathbf{f}) = \{f \in W_p^r[0, 1]: f(t_i) = f_i, i = 1, ..., n\}.$$

The following problem is considered by de Boor [6] and Fisher and Jerome [7]:

$$\inf\{\|f^{(r)}\|_p : f \in F_p^r(\mathbf{t}; \mathbf{f})\}. \tag{1.1}$$

Pinkus [13] discussed the problem

$$\inf_{\mathbf{t} \in \Xi_n} \inf\{\|f^{(r)}\|_p : f \in F_p^r(\mathbf{t}; \mathbf{f})\}, \tag{1.2}$$

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where, for convenience of exposition, it is supposed that the interpolation values $\{f_i\}_{i=1}^n$ satisfy the conditions

$$(f_i - f_{i-1})(f_{i+1} - f_i) < 0, \qquad i = 2, ..., n-1. \tag{1.3}$$

Pinkus proved that the solutions of the problem (1.2) must be of a particular form given by the solutions of (1.1), and the solutions must also be strictly monotone on $[t_i, t_{i+1}]$ for each $i = 1, ..., n-1$. Uniqueness of the solution to (1.2) is proved for $p = \infty$ by Pinkus [13], for $p = 2$ in case $r = 2$ by Marin [10] and in case $r = 3$ by Uluchev [16]. A generalization of the problem (1.2) allowing equalities in the assumption (1.3) is discussed by Bojanov [3]. Naidenov [11] gave an algorithm for the construction of the solutions to (1.2).

The purpose of this paper is to extend the problem (1.2) to the case of interpolation of mean values over intervals with fixed lengths; i.e., when we have interpolation values

$$\frac{1}{2h_i} \int_{t_i - h_i}^{t_i + h_i} f(x)\, dx, \qquad i = 1, ..., n,$$

where $\{h_i\}_{i=1}^n$ are fixed numbers. These numbers must be small enough to assure that the intervals $[t_i - h_i, t_i + h_i]$, $i = 1, ..., n$ are disjoint. We introduce some notation. Set

$$\Xi_n^{\mathbf{h}} = \{\mathbf{t} = (t_1, ..., t_n) : 0 \leqslant t_1 - h_1 < t_1 + h_2 < \cdots < t_n - h_n < t_n + h_n \leqslant 1\},$$

where $h_1, ..., h_n$ are fixed real number such that $h_i \in (0, 1/2(n-1))$, $i = 1, ..., n$, and $\mathbf{h} = \{h_i\}_{i=1}^n$. Let $e_1, ..., e_n$ be given positive numbers and

$$\mathbf{e} = \{(-1)^{i-1} e_i\}_{i=1}^n.$$

With each $\mathbf{t} \in \Xi_n^h$ and $\mathbf{e}$ we associate the set of functions

$$W_p^r(\mathbf{t}; \mathbf{h}; \mathbf{e}) = \left\{ f \in W_p^r[0, 1] : \frac{1}{2h_i} \int_{t_i - h_i}^{t_i + h_i} f(x)\, dx = (-1)^{i-1} e_i, \; i = 1, ..., n \right\}.$$

Set

$$h = \frac{\min\{e_i\}_{i=1}^n}{\max\{e_i\}_{i=1}^n (n-1)\, c(r)},$$

where $c(r) = (r+2) \sqrt[r]{2^{2r-1} r!}$. We impose the following restriction on $\{h_i\}_{i=1}^n$,

$$h_i < \frac{1}{16M} h^2, \qquad i = 1, ..., n, \qquad \text{for} \quad p \in (1, \infty), \tag{1.4}$$

and

$$h_i < \frac{1}{16M} h, \qquad i = 1, ..., n, \qquad \text{for} \quad p = \infty,$$

where M is the constant from the following theorem (see [17, Theorem 5.6]).

THEOREM A. *For $r \geq 2$ and $1 \leq p \leq \infty$, there is a constant M depending only on r such that for every $f \in W_p^r[0, 1]$*

$$u^k \|f^{(k)}\|_p \leq M(\|f\|_p + u^r \|f^{(r)}\|_p), \qquad k = 0, ..., r,$$

where $0 \leq u \leq 1$.

We consider the extremal problem

$$\inf_{\mathbf{t} \in \Xi_n^{\mathbf{h}}} \inf \{ \|f^{(r)}\|_p : f \in W_p^r(\mathbf{t}; \mathbf{h}; \mathbf{e}) \}. \tag{1.5}$$

We give a characterization of the solution of (1.5) and prove the uniqueness of the extremal function for $p = \infty$. When $h_i = 0$, $i = 1, ..., n$, this is the problem (1.2). The results in this paper are based on the total positivity structure of the problem. Some other extremal problems concerning interpolation in the mean we studied by Subbotin [15].

We assume that $n > r$. If $n \leq r$, then for any choice $\mathbf{t} \in \Xi_n^{\mathbf{h}}$, there exists a polynomial p of degree $r-1$ for which $(1/2h_i) \int_{t_i - h_i}^{t_i + h_i} p(x)\, dx = (-1)^{i-1} e_i$, $i = 1, ..., n$. Moreover $p^{(r)}(x) \equiv 0$ and our problem is trivially solved.

In Section 2 we discuss some auxiliary results. In Section 3 we consider the solving of (1.5).

2. SOME AUXILIARY RESULTS

In order to solve (1.5) we must consider first the problem

$$\inf \{ \|f^{(r)}\|_p : f \in W_p^r(\mathbf{t}; \mathbf{h}; \mathbf{e}) \}. \tag{2.1}$$

When $h_i = 0$, $i = 1, ..., n$, this is the problem (1.1). Let $\mathbf{t} \in \Xi_n^{\mathbf{h}}$, $n > r \geq 2$, and $\mathbf{e}$ be fixed. We introduce some preliminary definitions and properties. Set $\delta_i = \{t_i - h_i, t_i + h_i\}$, $i = 1, ..., n$, and suppose that $0 \leq t_1 - h_1 < t_1 + h_1 \leq t_2 - h_2 < t_2 + h_2 \cdots \leq t_n - h_n < t_n + h_n \leq 1$. Let Π_r denote the set of algebraic polynomials of degree less than or equal r. Denote by

$$l_i(f) = \frac{1}{2h_i} \int_{t_i - h_i}^{t_i + h_i} f(t)\, dt, \qquad i = 1, ..., n.$$

DEFINITION 1. Given pairs of points $\{\delta_j\}_{j=i}^{i+r}$ and an integrable function f, its rth "divided difference" over the pairs of points $\delta_i, ..., \delta_{i+r}$ is defined to be the coefficient of x^r in the unique polynomial $p(x) \in \Pi_r$ satisfying the conditions $l_j(p) = l_j(f)$, $j = I, ..., i+r$.

When $h_i = 0$, $i = 1, ..., n$, this is the usual divided difference. Let us explain the new "divided differences" a bit more. Denote by

$$U(0, 1, ..., r; t_i, ..., t_{i+r}) = \det\{t_j^k\}_{k=0,\, j=i}^{r\, i+r}$$

and

$$U(0, 1, ..., r-1, f; t_i, ..., t_{i+r}) = \det\{f_k(t_j)\}_{k=0,\, j=i}^{r\, i+r},$$

where $f_k(t) = t^k$, $k = 0, ..., r-1$ and $f_r(t) = f(t)$. Then, as is well known, the usual divided difference of f at the points $t_i, ..., t_{i+1}$ is simply

$$f[t_i, ..., t_{i+r}] = \frac{U(0, 1, ..., r-1, f; t_i, ..., t_{i+r})}{U(0, 1, ..., r-1, r; t_i, ..., t_{i+r})}.$$

The new "divided difference" is simply

$$f[\delta_i, ..., \delta_{i+r}] = \frac{\int_{t_i-h_i}^{t_i+h_i} \cdots \int_{t_{i+r}-h_{i+r}}^{t_{i+r}+h_{i+r}} U(0, 1, ..., r-1, f; s_i, ..., s_{i+r})\, ds_i \cdots ds_{i+r}}{\int_{t_i-h_i}^{t_i+h_i} \cdots \int_{t_{i+r}-h_{i+r}}^{t_{i+r}+h_{i+r}} U(0, 1, ..., r-1, r; s_i, ..., s_{i+r})\, ds_i \cdots ds_{i+r}}.$$

From this is easy to see some properties analogous to those of the usual divided differences (see Schumaker [14], Bojanov, Hakopian, and Sahakian [5]).

1.

$$f[\delta_i, ..., \delta_{i+r}] = \sum_{j=i}^{i+r} \frac{l_j(f)}{l_j(\omega'(\cdot, \tau^{\mathbf{j}}))}, \tag{2.2}$$

where $\omega'(\tau, \tau^{\mathbf{j}}) = (\tau - \tau_i^j) \cdots (\tau - \tau_{j-1}^j)(\tau - \tau_{j+1}^j) \cdots (\tau - \tau_{j+r}^j)$, $\tau_k^j \in (t_k - h, t_k + h)$, $k = i, ..., i+r$, $k \neq j$.

2. $f[\delta_i, ..., \delta_{i+r}]$ is the unique linear functional of the form $D_i(f) = \sum_{j=i}^{i+r} c_j l_j(f)$ satisfying the conditions

$$\begin{aligned} D_i(x^k) &= 0, \qquad k = 0, ..., r-1, \\ D_i(x^r) &= 1. \end{aligned} \tag{2.3}$$

3. For any $\xi \in (t_i - h_i, t_{i+r} + h_{i+r})$, $\xi \notin \{t_{i+1} - h_{i+1}, ..., t_{i+r} - h_{i+r}\}$, $\xi \notin \{t_i + h_i, ..., t_{i+r} + h_{i+r-1}\}$, there exists a constant $\alpha \in (0, 1)$ such that

3a. if $\xi \in (t_j - h_j, t_j + h_j)$ for some j, then

$$f[\delta_i, ..., \delta_{i+r}] = \alpha f[\delta_i, ..., \delta_{j-1}, \delta_\xi^a, \delta_{j+1}, ..., \delta_{i+r}] \\ + (1-\alpha) f[\delta_i, ..., \delta_{j-1}, \delta_\xi^b, \delta_{j+1}, ..., \delta_{i+r}],$$

where $\delta_\xi^a = \{t_j - h_j, \xi\}$ and $\delta_\xi^b = \{\xi, t_j + h_j\}$.

3b. If $\xi \in (t_j + h_j, t_{j+1} - h_{j+1})$ for some j, then

$$f[\delta_i, ..., \delta_{i+r}] = \alpha f[\delta_i, ..., \delta_{i+r-1}, \delta_\xi] + (1-\alpha) f[\delta_\xi, \delta_{i+1}, ..., \delta_{i+r}],$$

where $\delta_\xi = \{\xi - h_\xi, \xi + h_\xi\}$, and h_ξ is sufficiently small so that $t_j + h_j < \xi - h_\xi < \xi + h_\xi < t_{j+1} - h_{j+1}$.

Now we can define "B"-splines using the above definition and properties.

DEFINITION 2. The function

$$B_i(t) = (\cdot - t)_+^{r-1} [\delta_i, ..., \delta_{i+r}],$$

is called the rth order "B"-spline associated with the pairs of knots $\delta_i, ..., \delta_{i+r}$.

Here we also have properties analogous to those of the usual B-splines.

4. $B_i(x) = 0$ for $x \notin (t_i - h_i, t_{i+r} + h_{i+r})$ and $B_i(x) > 0$ for $x \in (t_i - h_i, t_{i+r} + h_{i+r})$.

Let $S^-_{(a,b)}(f)$ denote the number of sign changes of the function f on $[a, b]$. Similarly, for a vector $\beta \in R^m \backslash \{0\}$, $S^-(\beta)$ denotes the number of sign changes of the vector β.

5. Variation diminishing property.

$$S_R^-\left(\sum_{i=1}^{n-r} \beta_i B_i(t)\right) < S^-(\beta_1, ..., \beta_{n-r}).$$

6. Total positivity.

$$\Delta := \det \{B_{i_l}(y_{j_s})\}_{l=1, s=1}^{m, m} \geqslant 0,$$

for every choice of the points $y_1 < \cdots < y_n$ and integers $1 \leqslant m \leqslant n-r$, $1 \leqslant i_1 < \cdots < i_m \leqslant n-r$, $1 \leqslant j_1 < \cdots < j_m \leqslant n-r$. Moreover

$$\Delta > 0,$$

if and only if $y_{j_l} \in \operatorname{supp} B_{i_l}$, $l = 1, ..., m$. Here $\operatorname{supp} B_i = \{x : B_i(x) \neq 0\}$.

These properties we can analogously to the properties of the usual divided differences and B-splines and we omit it (see Schumaker [14], Bojanov, Hakopian, and Sahakian [5]).

Let us return to the problem in (2.1). For $f \in W_p^r[0, 1]$ we have

$$f(x) = \sum_{i=0}^{r-1} a_i x^i + \frac{1}{(r-1)!} \int_0^1 (x-y)_+^{r-1} f^{(r)}(y)\, dy. \tag{2.4}$$

Let $\mathbf{t} \in \Xi_n^{\mathbf{h}}$ be fixed. For $f \in W_p^r(\mathbf{t}; \mathbf{h}; \mathbf{e})$ set

$$E_i = f[\delta_i, ..., \delta_{i+r}], \qquad i = 1, ..., n-r.$$

Taking the "divided difference" at the pairs of points $\delta_i, ..., \delta_{i+r}$ of the function $f \in W_p^r(\mathbf{t}; \mathbf{h}; \mathbf{e})$ and using (2.4), (2.3), and the definition of the B_i, we obtain

$$E_i = \frac{1}{(r-1)!} \int_0^1 B_i(y)\, g(y)\, dy, \quad i = 1, ..., n-r,$$

where $g(y) = f^{(r)}(y)$. Problem (2.1) is equivalent to

$$\inf \left\{ \|g\|_p : \frac{1}{(r-1)!} \int_0^1 B_i(y)\, g(y)\, dy = E_i,\, i = 1, ..., n-r \right\}. \tag{2.5}$$

We may consider problem (2.5) as in the case for $h_i = 0$, $i = 1, ..., n$ (see de Boor [6]). For $p \in (1, \infty)$ (2.5) (and this (2.1)) has a unique solution of the form

$$g^*(y) = \left| \sum_{i=1}^{n-r} b_i B_i(y) \right|^{q-1} \operatorname{sign} \left(\sum_{i=1}^{n-r} b_i B_i(y) \right), \tag{2.6}$$

where $1/p + 1/q = 1$ and

$$E_i = \frac{1}{(r-1)!} \int_0^1 B_i(y)\, g^*(y)\, dy, \qquad i = 1, ..., n-r. \tag{2.7}$$

Equations (2.7) uniquely determine the coefficients $\{b_i\}_{i=1}^{n-r}$ in (2.6). To obtain the unique solution to (2.1) we write

$$f(x) = \sum_{i=0}^{r-1} a_i x^i + \frac{1}{(r-1)!} \int_0^1 (x-y)_+^{r-1} g^*(y)\, dy$$

and uniquely determine the $\{a_i\}_{i=0}^{r-1}$ so that $(1/2h_i) \int_{t_i-h_i}^{t_i+h_i} f(x)\, dx = (-1)^{i-1} e_i$, $i = 1, ..., r$. From (2.7) it follows that $f \in W_p^r(\mathbf{t}; \mathbf{h}; \mathbf{e})$.

Further we will need the following two lemmas.

LEMMA 1. *Let* $P(x)=\sum_{i=1}^{n-r} b_i B_i(x)$, *where the* $\{b_i\}_{i=1}^{n-r}$ *are determined by*

$$E_i=\frac{1}{(r-1)!}\int_0^1 B_i(y)\,|P(y)|^{q-1}\operatorname{sign}(P(y))\,dy, \qquad i=1, ..., n-r.$$

$P(x)$ *has exactly* $n-r-1$ *sign changes on* $(0, 1)$ *and does not vanish on any subinterval of* $[0, 1]$.

Proof. We remark that $E_i E_{i+1}<0$ for $i=1, ..., n-r-1$. Indeed from (2.2) it follows that

$$\begin{aligned}\operatorname{sign} E_i &= \operatorname{sign}\sum_{j=i}^{i+r}\frac{(-1)^{j-1}e_j}{l_j(\omega'(\cdot,\tau^{\mathbf{j}}))}\\ &=\operatorname{sign}\sum_{j=i}^{i+r}(-1)^{j-1}(-1)^{r+i-j}=(-1)^{r+i-1}.\end{aligned}$$

For the proof that $p(x)$ has exactly $n-r-1$ sign changes on (0, 1) we may proceed analogously to that in Pinkus [13] (see Proposition 2.1) using properties 4–6 of the new “B”-splines, the variations diminishing property of Total Positivity of matrices (see Karlin [8]) and the fact that $E_i E_{i+1}<0$ for $i=1, ..., n-r-1$.

We will show that $P(x)$ does not vanish on any subinterval of [0, 1]. Since $P(x)$ has exactly $n-r-1$ sign changes and since we have the inequality

$$S^-(b_1, ..., b_{n-r})\leqslant n-r-1,$$

it follows that $S^-(b_1, ..., b_{n-r})=n-r-1$ and

$$b_i(-1)^{(i-1+r)}>0, \qquad i=1, ..., n-r. \tag{2.8}$$

Using the definition of $B_i(x)$ and (2.2) we get

$$B_i^{(r)}(x)=(-1)^r\,(r-1)!\sum_{j=i}^{i+r}\frac{(t_j+h_j-x)_+^0-(t_j-h_j-x)_+^0}{2h_j l_j(\omega'(\cdot\,\tau^{\mathbf{j}}))}.$$

Therefore

$$B_i^{(r)}(x)(-1)^{i+j}>0 \qquad \text{for} \quad x\in(t_j-h_j,\, t_j+h_j), \qquad j=i, ..., i+r, \tag{2.9}$$

and

$$B_i^{(r)}(x)=0 \qquad \text{for} \quad x\in(t_j+h_j,\, t_{j+1}-h_{j+1}), \qquad j=i, ..., i+r-1; \tag{2.10}$$

i.e., the rth derivative of $B_i(x)$ strictly alternates in sign as we pass from (t_j-h_j, t_j+h_j) to $(t_{j+1}-h_{j+1}, t_{j+1}+h_{j+1})$ for $j=i, ..., i+r-1$. By (2.8) and (2.9) for $x\in(t_j-h_j, t_j+h_j)$ we obtain

$$P^{(r)}(x)=(-1)^{j-1+r}\sum_{i=1}^{n-r}|b_iB_i^{(r)}(x)|.$$

Since $b_i\neq 0$ for all i, and $B_i^{(r)}(x)\not\equiv 0$ on (t_j-h_j, t_j+h_j) for some i, it follows that

$$(-1)^{j-1+r}P^{(r)}(x)>0 \quad \text{on} \quad (t_j-h_j, t_j+h_j) \qquad \text{for} \quad j=1, ..., n. \tag{2.11}$$

Therefore $P(x)$ does not vanish on the subintervals (t_j-h_j, t_j+h_j) for $j=1, ..., n$.

It remains to show that $P(x)\not\equiv 0$ on $[t_j+h_j, t_{j+1}-h_{j+1}]$ for $j=1, ..., n-1$. Assume the contrary, that $P(x)\equiv 0$ for $x\in[a, b]\subseteq[t_k+h_k, t_{k+1}-h_{k+1}]$ for some $k\in\{1, ..., n-1\}$. If we suppose that $P(y)$ has i sign changes on $(0, a)$, then $P(x)$ has at least $n-r-2-i$ sign changes on $(b, 1)$ since $P(x)$ has exactly $n-r-1$ sign changes on $(0, 1)$. The points, 0, a, b and 1 are zeros of multiplicity r. Applying Rolle's Theorem r times we obtain that $P^{(r)}(y)$ has at least $i+r$ sign changes on $(0, a)$ and at least $n-i-2$ sign changes on $(b, 1)$. It follows by (2.11) that when we pass from $(0, a)$ to $(b, 1)$ we obtain one more sign change; i.e., $P^{(r)}(y)$ has at least $n+r-1$ sign changes on $(0, 1)$. From (2.10) and (2.11) we see that $P^{(r)}(x)\equiv 0$ on $[t_j+h_j, t_{j+1}-h_{j+1}]$ for $j=1, ..., n-1$ and $P^{(r)}(x)$ has exactly $n-1$ sign changes on $(0, 1)$. Thus we obtain a contradiction and Lemma 1 is proved.

Denote by $f(\mathbf{t};\,\cdot\,)$ the unique solution of problem (2.1) for $p\in(1, \infty)$.

LEMMA 2. *For fixed* $\mathbf{t}\in\Xi_n^h$, $\|f^{(r)}(\mathbf{t};\,\cdot\,)\|_p$ *is a strictly increasing function of* $e_1, ..., e_n$.

The proof proceeds analogously to that in Naidenov [11] (see Lemma 3) using properties 1–6 of the new "divided differences" and "B"-splines. We omit it.

Now we consider (2.1) for $p=\infty$. By (2.5) we reduce the problem of existence to a moment problem. It is standard (e.g., Hobby and Rice [12]) to show that there exists a g, $|g(x)|=\|g\|_\infty$ for all x, with at most $n-r-1$

jumps which gives the appropriate moments. The next theorem gives one solution to (2.1) for $p=\infty$ in a particular form.

THEOREM 1. *The set* $W^r_\infty(\mathbf{t}; \mathbf{h}; \mathbf{e})$ *contains a perfect spline* $P(x)$ *of degree* r *with exactly* $n-r-1$ *knots; i.e., a function* P *of the form*

$$P(x)=\sum_{i=0}^{r-1} a_i x^i+\frac{R}{r!}\left[x^r+2\sum_{i=1}^{n-r-1}(-1)^i(x-\xi_i)^r_+\right], \tag{2.12}$$

where $0<\xi_1<\cdots<\xi_{n-r-1}<1$. *Moreover this perfect spline is a solution of problem* (2.1) *for* $p=\infty$.

The existence of $P(x)$ can be proved by the same argument as after (2.7). The minimality property follows easily in many ways. For example, we assume that $\|f^{(r)}\|_\infty<\|P^{(r)}\|_\infty$ for some $f\in W^r_\infty(\mathbf{t}; \mathbf{h}; \mathbf{e})$. Then by the interpolation conditions we have $\int_0^1 B_i(t)(P^{(r)}(t)-f^{(r)}(t))\,dt=0$, $i=1, ..., n-r$. But $P^{(r)}(x)-f^{(r)}(x)$ has at most $n-r-1$ sign changes on (0, 1) and does not vanish identically on any subinterval of [0, 1], and thus cannot be orthogonal to the WT-system $\{B_i\}_{i=1}^{n-r}$.

Let $P_0(x)$ be the unique perfect spline with $n-r-1$ knots and $\{x_i^0\}_{i=1}^n$ be the unique set of points satisfying the following conditions

$$\begin{aligned} &P_0(x_i^0)=(-1)^{i-1}(4/3)\max\{e_i\}_{i=1}^n, && i=1, ..., n,\\ &P_0'(x_i^0)=0, && i=2, ..., n-1, \end{aligned} \tag{2.13}$$

where $x_1^0=0$, $x_n^0=1$. The existence and uniqueness of such a perfect spline P_0 and such a set of points $\{x_i^0\}_{i=1}^n$ are proved in [1]. It holds $\|P_0\|_\infty=(4/3)\max\{e_i\}_{i=1}^n$.

We will frequently use the following property of the perfect spline P of degree r with $n-r-1$ knots: If P changes sign $n-1$ times in (0, 1), then P has exactly $n-1$ zeros and P' has exactly $n-2$ zeros. This follows by an application of Rolle's Theorem (see, for example [5]).

LEMMA 3. *If* P_0 *is the perfect spline from* (2.13) *and* $R_0=\|P_0^{(r)}\|_\infty$, *then*

$$R_0\leqslant(4/3)\max\{e_i\}_{i=1}^n(n-1)^r\,2^{2r-1}(r+2)^r\,r!. \tag{2.14}$$

Proof. Define $I=\max\{x_{j+1}^0-x_j^0\}_{i=1}^{n-1}=x_{j_0+1}^0-x_{j_0}^0$, $x_1^0=0$, $x_n^0=1$. We have

$$I\geqslant\frac{1}{n-1}, \tag{2.15}$$

because these points partition the whole interval from 0 to 1. Let ξ_i^0, $i=1, ..., n-r-1$ be the knots of P_0. P_0 and P_0' are perfect splines with a maximal number of zeros. For such perfect splines its zeros and knots satisfy the so-called interlacing conditions; i.e.,

$$x_i^0 < \xi_i^0 < x_{i+r+1}^0, \qquad i = 1, ..., n-r-1.$$

Therefore between $x_{j_0}^0$ and $x_{j_0+1}^0$ we have at most $r+1$ knots of $P_0(x)$. Suppose that k knots of $P_0(x)$ lie between $x_{j_0}^0$ and $x_{j_0+1}^0$. Denote them by $\eta_1, ..., \eta_k$ and $I_s = \eta_{s+1} - \eta_s$, $s = 0, ..., k$, $(\eta_0 = x_{j_0}^0,\ \eta_{k+1} = x_{j_0+1}^0)$. Between two neighboring knots η_s, η_{s+1} we have that $P_0(x)$ is a polynomial of degree r, where $\|P_0^{(r)}\|_\infty / r!$ is the absolute value of the coefficient of x^r. The Markov inequality (see [9]) asserts that for every polynomial Q of degree r on $[-1, 1]$

$$\|Q^{(i)}\|_\infty \leqslant \|T_r^{(i)}\|_\infty \|Q\|_\infty, \qquad i = 1, ..., r,$$

where T_r is the rth degree classical Chebyshev polynomial. Applying this inequality for $i = r$, $Q = P_0$ on $[\eta_s, \eta_{s+1}]$, we obtain

$$\|P_0\|_\infty \geqslant \max_{x \in (\eta_s, \eta_{s+1})} |P_0(x)| \geqslant \frac{R_0 I_s^r}{2^{2r-1} r!}, \qquad s = 0, ..., k, \tag{2.16}$$

where $R_0 = \|P_0^{(r)}\|_\infty$. From (2.15) and (2.16) we get

$$(k+1) \sqrt[r]{\frac{\|P_0\|_\infty 2^{2r-1} r!}{R_0}} \geqslant I_0 + \cdots + I_k = x_{j_0+1}^0 - x_{j_0}^0 \geqslant \frac{1}{n-1}. \tag{2.17}$$

Therefore, since $k+1 \leqslant r+2$, and in view of (2.17), we obtain

$$R_0 \leqslant \|P_0\|_\infty (n-1)^r\, 2^{2r-1} (r+2)^r\, r!.$$

Therefore (2.14) holds.

Now we will show some properties of P_0, which we will need later. Since $P_0^{(j)}(x)$ vanishes on $(0, 1)$ it easily follows that $\|P_0^{(j)}\|_p \leqslant \|P_0^{(j+1)}\|_p$ for $j = 0, ..., r-1$ and any $p \in [1, \infty]$. Therefore $\|P_0\|_\infty \leqslant \|P_0^{(r)}\|_\infty$. Then we can set $u = \sqrt[r]{\|P_0\|_\infty / \|P_0^{(r)}\|_\infty}$ and apply Theorem A for P_0, $k = 1$ and $p = \infty$. We obtain

$$\|P_0'\|_\infty \leqslant 2M \|P_0\|_\infty^{(r-1)/r} \|P_0^{(r)}\|_\infty^{1/r}. \tag{2.18}$$

By (2.13), (2.14), and (2.18), it follows that

$$\|P_0'\|_\infty \leqslant M(8/3) \max\{e_i\}_{i=1}^n c(r)(n-1). \tag{2.19}$$

P_0 has exactly $n-1$ zeros (see [1]) $\{\tau_i^0\}_{i=1}^{n-1} \subset (0, 1)$. Using Taylor's formula we see that

$$|P_0(x_i^0)| \leqslant (x_i^0 - \tau_{i-1}^0) \, \|P_0'\|_\infty, \qquad i = 2, ..., n,$$

$$|P_0(x_i^0)| \leqslant (\tau_i^0 - x_i^0) \, \|P_0'\|_\infty, \qquad i = 1, ..., n-1.$$

Applying (2.19) in the last inequalities and by the choice of h_i in (1.4) we obtain

$$\begin{aligned} x_i^0 - \tau_{i-1}^0 &> \frac{1}{2M(n-1)\, c(r)} > 2h_i, \qquad i = 2, ..., n \\ \tau_i^0 - x_i^0 &> \frac{1}{2M(n-1)\, c(r)} > 2h_i, \qquad i = 1, ..., n-1. \end{aligned} \tag{2.20}$$

Therefore there exist points

$$t_1^0 = h_1, \qquad t_i^0 \in (\tau_{i-1}^0, \tau_i^0), \qquad i = 2, ..., n-1, \qquad t_n^0 = 1 - h_n,$$

such that

$$\begin{aligned} P_0(t_i^0 - h_i) &= P_0(t_i^0 + h_i) \qquad & i = 2, ..., n-1, \\ \frac{1}{2h_i} \int_{t_i^0 - h_i}^{t_i^0 + h_i} P_0(x)\, dx &= (-1)^{i-1} e_i^0, \qquad & i = 1, ..., n, \\ (t_i^0 - h_i, t_i^0 + h_i) &\subset (\tau_{i-1}^0, \tau_i^0), \qquad & i = 1, ..., n, \end{aligned} \tag{2.21}$$

where $\tau_0^0 = 0$ and $\tau_n^0 = 1$. We have $e_i^0 > 0$, since $(-1)^{i-1} P_0(x) > 0$ for $t \in (\tau_{i-1}^0, \tau_i^0)$, $i = 1, ..., n$. By (2.13) we see that

$$e_i^0 < (4/3) \max\{e_i\}_{i=1}^n, \qquad i = 1, ..., n. \tag{2.22}$$

Using (2.21) we get

$$|P_0(x_i^0)| \leqslant |P_0(t_i^0 - h_i)| + 2h_i \, \|P_0'\|_\infty, \qquad i = 1, ..., n.$$

Hence by (2.13), (2.19), (2.21), and (1.4), we obtain

$$e_i^0 > |P_0(t_i^0 - h_i)| > \max\{e_i\}_{i=1}^n, \qquad i = 1, ..., n. \tag{2.23}$$

3. SOLVING (1.5)

3.1. *Case* $p \in (1, \infty)$

THEOREM 2. *Let* $p \in (1, \infty)$ *and let* $f^* \in W_p^r[0, 1]$ *be a solution of the problem* (1.5). *There exists* $\mathbf{t}^* \in \Xi_n^{\mathbf{h}}$ *such that* $f^* \in W_p^r(\mathbf{t}^*; \mathbf{h}; \mathbf{e})$. f^* *satisfies the following conditions*:

$$f^{*(r)}(y) = \left| \sum_{i=1}^{n-r} b_i^* B_i(y) \right|^{q-1} \operatorname{sign} \left(\sum_{i=1}^{n-r} b_i^* B_i(y) \right), \tag{3.1.1}$$

where $1/p + 1/q = 1$ *and* $B_i(y) = (\cdot - y)^{r-1} [\delta_i^*, ..., \delta_{i+r}^*]$, *where* $\delta_j^* = \{t_j^* - h_j,\ t_j^* + h_j\}$, $j = 1, ..., n$ *and* $\{b_i^*\}_{i=1}^{n-r}$ *are determined by the system of equations*

$$\int_0^1 b_i(y)\, f^{*(r)}(y)\, dy = f^*[\delta_i^*, ..., \delta_{i+r}^*], \qquad i = 1, ..., n-r; \tag{3.1.2}$$

and

$$\begin{aligned} f^*(t_i^* + h_i) &= f^*(t_i^* - h_i), \qquad i = 2, ..., n-1, \\ t_1^* = h_1, &\qquad t_n^* = 1 - h_n. \end{aligned} \tag{3.1.3}$$

Proof. Let the function f^* and the points $\mathbf{t}^* = \{t_i^*\}_{i=1}^n$ solve (1.5). Since f^* must also solve (2.1) for $\mathbf{t}^*$ and $p \in (1, \infty)$, it follows that f^* necessarily satisfies (3.1.1) and (3.1.2).

It remains to prove (3.1.3).

Let $P_0(x)$ be the perfect spline from Lemma 3. Let the points $\{t_i^0\}_{i=1}^n$ and the values $\{e_i^0\}_{i=1}^n$ be chosen as in (2.21). Denote by g and g^* the unique solutions of (2.1) with interpolation points $\{t_i^0\}_{i=1}^n$ and interpolation values respectively $\{(-1)^{i-1} e_i^0\}_{i=1}^n$ and $\{(-1)^{i-1} e_i\}_{i=1}^n$. From (2.23) we have $e_i^0 > e_i$, $i = 1, ..., n$. Therefore by Lemma 2 we have $\|g^{*(r)}\|_p < \|g^{(r)}\|_p$. On the other hand $\|g^{(r)}\|_p < \|P_0^{(r)}\|_p$, since from (2.21) we have that P_0 satisfies the interpolation conditions in (2.1). Therefore if $f^* \in W_p^r(\mathbf{t}^*; \mathbf{h}; \mathbf{e})$ is a solution of (1.5), then the following inequalities hold

$$\|f^{*(r)}\|_p \leqslant \|g^{*(r)}\|_p < \|g^{(r)}\|_p < \|P_0^{(r)}\|_p. \tag{3.1.4}$$

Since P_0 is a perfect spline, $|P_0^{(r)}(x)| = R_0$ for all x and thus $\|P_0^{(r)}\|_p = R_0$, trivially. Therefore by (3.1.4), (2.14) and the last equality

$$\|f^{*(r)}\|_p \leqslant (4/3) \max\{e_i\}_{i=1}^n (n-1)^r\, 2^{2r-1} (r+2)\, r!. \tag{3.1.5}$$

By the character of the data f^* has at least $n-1$ distinct zeros on $(0, 1)$. From Lemma 1 $f^{*(r)}$ does not vanish on any subinterval of $(0, 1)$ and has exactly $n-r-1$ sign changes. By Rolle's Theorem it follows that f^* has exactly $n-1$ simple zeros $\tau_1 < \cdots < \tau_{n-1}$ on $(0, 1)$. Therefore $\|f^*\|_p/\|f^{*(r)}\|_p \leqslant 1$. Then using (3.1.5) we apply Theorem A for f^*, $p \in (1, \infty)$, $k=2$ and $u = \sqrt[r]{\|f^*\|_p/f^{*(r)}\|_p}$ to get

$$\|f^{*\prime\prime}\|_p \leqslant 2M \|f^*\|_p^{(r-2)/r} (c(r)(n-1))^2 ((4/3) \max\{e_i\}_{i=1}^n)^{2/r}. \tag{3.1.6}$$

Applying the inequalities

$$\|f^*\|_p^{(r-2)/r} \leqslant \max_{x \in [0,1]} |f^*(x)|^{(r-2)/r},$$

and

$$\|f^*\| := \max_{x \in [0,1]} |f^*(x)| > \min\{e_i\}_{i=1}^n,$$

in (3.1.6), we get

$$\begin{aligned}\|f^{*\prime\prime}\|_p &\leqslant 2M \|f^*\| (c(r)(n-1))^2 \left((4/3) \frac{\max\{e_i\}_{i=1}^n}{\min\{e_i\}_{i=1}^n}\right)^{2/r} \\ &\leqslant (8/3)\, M \|f^*\| (c(r)(n-1))^2 \frac{\max\{e_i\}_{i=1}^n}{\min\{e_i\}_{i=1}^n}.\end{aligned} \tag{3.1.7}$$

Let $x_i \in (\tau_{i-1}, \tau_i)$, $i = 2, ..., n-1$, be the extremal points of $f^*(x)$. Since $f^{*\prime}$ has exactly $n-2$ zeros on $(0, 1)$, $f^*(x)$ is monotone on $[0, \tau_1]$ and $[\tau_{n-1}, 1]$. Then

$$\|f^*\| = |f^*(x_{i_0})|, \tag{3.1.8}$$

where $i_0 \in \{1, ..., n\}$, and $x_1 = 0$, $x_n = 1$.

First we will show that

$$\begin{gathered} f^*(t^*_{i_0} - h_{i_0}) = f^*(t^*_{i_0} + h_{i_0}) \quad \text{if } i_0 \in \{2, ..., n-1\}, \\ t^*_1 = h_1 \quad \text{if } i_0 = 1 \qquad \text{and} \qquad t^*_n = 1 - h_n \quad \text{if } i_0 = n. \end{gathered} \tag{3.1.9}$$

Suppose that $i_0 \in \{2, ..., n-1\}$. Using the properties of the usual divided difference we get

$$f^*[x_{i_0}, \tau_{i_0-1}, \tau_{i_0}] = \int_{\tau_{i_0-1}}^{\tau_{i_0}} (x-t)_+ [x_{i_0}, \tau_{i_0-1}, \tau_{i_0}] f^{*\prime\prime}(t)\, dt, \tag{3.1.10}$$

$$|f^*[x_{i_0}, \tau_{i_0}, \tau_{i_0}]| = \frac{|f^*(x_{i_0})|}{(\tau_{i_0} - x_{i_0})(x_{i_0} - \tau_{i_0} - \tau_{i_0-1})}, \tag{3.1.11}$$

$$(\tau_{i_0} - \tau_{i_0} - 1)(x-t)_+ [x_{i_0}, \tau_{i_0-1}, \tau_{i_0}] < 1. \tag{3.1.12}$$

Applying Hölder's inequality in (3.1.10) and in view of (3.1.11) and (3.1.12) we obtain

$$\begin{aligned} \frac{|f^*(x_{i_0})|}{(\tau_{i_0} - \tau_{i_0-1})(x_{i_0} - \tau_{i_0-1})} &< \frac{|f^*(x_{i_0})|}{(\tau_{i_0} - x_{i_0})(x_{i_0} - \tau_{i_0-1})} \\ &\leqslant \frac{1}{\tau_{i_0} - \tau_{i_0-1}} \left(\int_{\tau_{i_0-1}}^{\tau_{i_0}} 1\, dt \right)^{1/q} \|f^{*\prime\prime}\|_p \\ &< \frac{\|f^{*\prime\prime}\|_p}{\tau_{i_0} - \tau_{i_0-1}}. \end{aligned} \tag{3.1.13}$$

From (3.1.7), (3.1.8), (3.1.13), and the choice of h_{i_0} in (1.4) we get

$$\begin{aligned} x_{i_0} - \tau_{i_0-1} &> \frac{3}{8M((n-1)\, c(r))^2} \frac{\min\{e_i\}_{i=1}^n}{\max\{e_i\}_{i=1}^n} \\ &> \frac{3}{8M((n-1)\, c(r))^2} \left(\frac{\min\{e_i\}_{i=1}^n}{\max\{e_i\}_{i=1}^n} \right)^2 \\ &> 4h_{i_0}. \end{aligned}$$

Analogously we see that $\tau_{i_0} - x_{i_0} > 4h_{i_0}$. Thus we obtain $\tau_{i_0} - \tau_{i_0-1} > 8h_{i_0}$ if $i_0 \in \{2, ..., n-1\}$. In the cases $i_0 = 1$ and $i_0 = n$ we consider $f^*[0, \tau_1, \tau_2]$ and $f^*[\tau_{n-2}, \tau_{n-1}, 1]$, respectively. Then analogously as above we show that

$$\tau_1 > 4h_1 \qquad \text{and} \qquad 1 - \tau_{n-1} > 4h_n. \tag{3.1.14}$$

Therefore we can find a point $\eta_{i_0} \in (\tau_{i_0-1}, \tau_{i_0})$, such that

$$\frac{1}{2h_{i_0}} \int_{\eta_{i_0} - h_{i_0}}^{\eta_{i_0} + h_{i_0}} f^*(x)\, dx = (-1)^{i_0 - 1} d_{i_0},$$

and

$$\begin{aligned} &f^*(\eta_{i_0} - h_{i_0}) = f^*(\eta_{i_0} + h_{i_0}), \quad \text{if } i_0 \in \{2, ..., n-1\}, \\ &\qquad \eta_1 = h_1 \quad \text{if } i_0 = 1, \qquad \text{and} \qquad \eta_n = 1 - h_n \quad \text{if } i_0 = n. \end{aligned}$$

Then η_{i_0} is the extremal point for the function $F(x)=\int_{x-h_{i_0}}^{x+h_{i_0}} f^*(t)\,dt$ in $(\tau_{i_0-1}, \tau_{i_0})$ if $i_0 \in \{2, ..., n-1\}$. In the cases $i_0=1$ and $i_0=n$, $F(x)$ is strictly monotone on $[h_1, \tau_1]$ and $[\tau_{n-1}, 1-h_n]$, respectively. Then $|F(x)|$ attains maximal values for $x=\eta_1$ and $x=\eta_n$, respectively.

Now if we assume that (3.1.9) does not hold, then

$$d_{i_0} > e_{i_0}. \tag{3.1.15}$$

Consider problem (2.1) for $p \in (1, \infty)$ at the point

$$\mathbf{t}=(t_1^*, ..., t_{i_0-1}^*, \eta_{i_0}, t_{i_0+1}^*, ..., t_n^*),$$

first with

$$\mathbf{e}=\{(-1)^{i-1} e_i\}_{i=1}^n$$

and second with

$$\mathbf{e}=(e_1, ..., (-1)^{i_0-2} e_{i_0-1}, (-1)^{i_0-1} d_{i_0}, (-1)^{i_0} e_{i_0+1}, ... (-1)^{n-1} e_n).$$

There are unique solutions to these two problems which we denote by g and g^*, respectively. From lemma 2 and (3.1.15) it follows that $\|g^{(r)}\|_p < \|g^{*(r)}\|_p$. On the other hand $\|g^{*(r)}\|_p < \|f^{*(r)}\|_p$, since f^* satisfies the same interpolation conditions at the points $t_1^*, ..., t_{i_0-1}^*$, η_{i_0}, $t_{i_0+1}^*, ..., t_n^*$ as g. Therefore $\|g^{(r)}\|_p < \|f^{*(r)}\|_p$, which contradicts the minimality property of f^*.

We will estimate $\|f^*\|$ and $\|f^{*''}\|_p$. From the Mean Value Theorem we have $f^*(y_{i_0})=(-1)^{i_0=1} e_{i_0}$ for some point $y_{i_0} \in (t_{i_0}^*-h_{i_0}, t_{i_0}^*+h_{i_0})$. First suppose that $i_0 \in \{2, ..., n-1\}$. By (3.1.9) we have that $x_{i_0} \in (t_{i_0}^*-h_{i_0}, t_{i_0}^*+h_{i_0})$, $f^*(t_{i_0}^*-h_{i_0})=f^*(t_{i_0}^*+h_{i_0})$ and $|f^*(t_{i_0}^*-h_{i_0})| < |f^*(y_{i_0})| = e_{i_0}$. Then similarly to (3.1.10), (3.1.11), (3.1.12), and (3.1.13) we get

$$\begin{aligned}
&\frac{|f^*(x_{i_0})|-|f^*(t_{i_0}^*-h_{i_0})|}{(2h_{i_0})^2}\\
&\quad< |f^*[t_{i_0}^*-h_{i_0}, x_{i_0}, t_{i_0}^*+h_{i_0}]|\\
&\quad\leqslant \frac{1}{2h_{i_0}} \int_{t_{i_0}^*-h_{i_0}}^{t_{i_0}^*+h_{i_0}} 2h_{i_0}(x-t)_+ [t_{i_0}^*-h_{i_0}, x_{i_0}, t_{i_0}^*+h_{i_0}]\, |f^{*''}(t)|\, dt\\
&\quad\leqslant \frac{1}{2h_{i_0}} \|f^{*''}\|_p.
\end{aligned} \tag{3.1.16}$$

Therefore

$$\|f^*\| = |f^*(x_{i_0})| < |f^*(t^*_{i_0} - h_{i_0})| + 2h_{i_0} \|f^{*\prime\prime}\|_p$$
$$< e_{i_0} + 2h_{i_0} \|f^{*\prime\prime}\|_p. \tag{3.1.17}$$

Applying (3.17) and (1.4) in (3.1.17), we get

$$\|f^*\| < \max\{e_i\}_{i=1}^n + \tfrac{1}{3} \|f^*\|. \tag{3.1.18}$$

Therefore $\|f^*\| < \frac{3}{2} \max\{e_i\}_{i=1}^n$. If $i_0 = 1$, then we consider $f[0, y_1, 2y_1]$. From (3.1.14), $y_1 < 2y_1 < 4h_1 < \tau_1$. Therefore $|f(0)| - 2|f(y_1)|/2y_1^2 < |f[0, y_1, 2y_1]|$. Similarly to (3.1.16), (3.1.17), and (3.1.18) we obtain $|f(0)| = \|f^*\| < 3 \max\{e_i\}_{i=1}^n$. Analogously we get $|f(1)| = \|f^*\| < 3 \max\{e_i\}_{i=1}^n$ if $i_0 = n$. Therefore $\|f^*\| < 3\max\{e_i\}_{i=1}^n$. Applying this inequality in (3.1.7) we get

$$\|f^{*\prime\prime}\|_p \leqslant 8M(c(r)(n-1))^2 \frac{(\max\{e_i\}_{i=1}^n)^2}{\min\{e_i\}_{i=1}^n}. \tag{3.1.19}$$

By the Mean Value Theorem there are points $y_i \in (t_i^* - h_i, t_i^* + h_i)$, $i = 1, ..., n$ such that $f^*(y_i) = (-1)^{i-1} e_i$, $i = 1, ..., n$. Analogously to (3.1.13) we obtain

$$\frac{\min\{e_i\}_{i=1}^n}{(\tau_i - \tau_{i-1})^2} \leqslant \frac{|f^*(y_i)|}{(\tau_i - \tau_{i-1})^2} < |f^*[\tau_{i-1}, y_i, \tau_i]| < \frac{\|f^{*\prime\prime}\|_p}{\tau_i - \tau_{i-1}},$$

for $i = 1, ..., n$. Then from (3.1.19), (1.4) and the last inequalities we see that

$$\tau_i - \tau_{i-1} > \frac{(\min\{e_i\}_{i=1}^n)^2}{8M(\max\{e_i\}_{i=1}^n (n-1)\, c(r))^2} > 2h_i, \qquad i = 1, ..., n.$$

Therefore we can find points $\eta_i \in (\tau_{i-1}, \tau_i)$, $i = 1, ..., n$, such that

$$\frac{1}{2h_i} \int_{\eta_i - h_i}^{\eta_i + h_i} f^*(x)\, dx = (-1)^{i-1} d_i, \qquad i = 1, ..., n,$$

and

$$f^*(\eta_i - h_i) = f^*(\eta_i + h_i), \qquad \text{for} \quad i = 2, ..., n-1,$$

$$\eta_1 = h_1, \qquad \eta_n = 1 - h_n.$$

Then η_i are the extremal points for the function $F(x) = \int_{x-h_i}^{x+h_i} f^*(t)\, dt$ in (τ_{i-1}, τ_i) for $i = 2, ..., n-1$. For $i = 1$ and $i = n$, $F(x)$ is strictly monotone on $[h_1, \tau_1]$ and $[\tau_{n-1}, 1-h_n]$, respectively. Then $|F(x)|$ attains maximal values for $x=\eta_1$ and $x=\eta_n$, respectively. Therefore

$$d_i \geqslant e_i, \qquad i=1, ..., n. \tag{3.1.20}$$

If we assume that (3.1.3) does not hold, then we have at least one strong inequality in (3.1.20). Further we obtain a contradiction with the minimality property of f^*, using Lemma 2 and similar arguments as in the proof of (3.1.9). The theorem is proved.

3.2. *Case* $p=\infty$

Denote by $P_{r, n-r-1, \mathbf{2h}}$ the set of perfect splines; i.e., functions of the form (2.12), with $n-1$ distinct zeros $\tau_1, ..., \tau_{n-1}$ satisfying $\tau_i - \tau_{i-1} > 2h_i$, $i=2, ..., n-1$, $\tau_1 > 2h_1$, $1-\tau_{n-1} > 2h_n$, and $\|P^{(r)}\|_\infty = 1$.

THEOREM 3. *There exists a unique perfect spline* $P \in P_{r, n-r-1, \mathbf{2h}}$, *a unique set of points* $\mathbf{t}^* \in \Xi_n^{\mathbf{h}}$ *and a positive number* R *such that*

$$\frac{R}{2h_i} \int_{t_i^*-h_i}^{t_i^*+h_i} P(x)\, dx = (-1)^{i-1} e_i, \qquad i=1, ..., n.$$

P is uniquely characterized by the conditions

$$P(t_i^* - h_i) = P(t_i^* + h_i), \qquad i=2, ..., n-1.$$

The points $\{t_i^*\}_{i=1}^n$ *satisfy*

$$0 = t_1^* - h_1 < t_1^* + h_1 < \tau_1 < t_2^* - h_2 < \cdots < \tau_{n-1} < t_n^* - h_n < t_n^* + h_n = 1,$$

where $\{\tau_i\}_{i=1}^{n-1}$ *are the zeros of* P. *Moreover,* R *is a strictly increasing function of each* e_i, $i=1, ..., n$.

Proof. Using a similar approach to that in Theorem 3.1 [4] we will prove the existence of the desired spline P by a continuous deformation of an initial perfect spline P_0, described by a system of nonlinear equations. We choose P_0 to be the perfect spline from Lemma 3. This will assume that the intervals over which we integrate are disjoint at each step. Let the points $\{t_i^0\}_{i=1}^n$ and the values $\{e_i^0\}_{i=1}^n$ be those from (2.21).

Put $e_j(s) = e_j^0 + s(e_j - e_j^0)$, $j=1, ..., n$, $s \in [0, 1]$. For $s \in [0, 1]$ we will construct a function $P(s; \cdot)$ of the form (2.12) with parameters

$$a_0^s, ..., a_{r-1}^s, t_2^s, ..., t_{n-1}^s, \tau_1^s, ..., \tau_{n-1}^s, \xi_1^s, ..., \xi_{n-r-1}^s, R_s,$$

such that

$$\begin{aligned} P(s, \tau_i^s) &= 0, & i &= 1, ..., n-1, \\ P(t_i^s - h_i) - P(t_i^s + h_i) &= 0, & i &= 2, ..., n-1, \\ \frac{R_s}{2h_i} \int_{t_i^s - h_i}^{t_i^s + h_i} P(s, t)\, dt &= (-1)^{i-1} e_i(s), & i &= 1, ..., n, \end{aligned} \tag{3.2.1}$$

where $t_1^s = h_1$ and $t_n^s = 1 - h_n$. Denote by $\Delta(s)$ the Jacobian of (3.2.1) with respect to $\{\tau_i^s\}_{i=1}^{n-1}$, $\{t_i^s\}_{i=2}^{n-1}$, R_s, $\{a_i^s\}_{i=0}^{r-1}$ and $\{\xi_i^s\}_{i=1}^{n-r-1}$. Here $P(0, x) = P_0(x)/\|P_0^{(r)}\|_\infty$ and $R_0 = \|P_0^{(r)}\|_\infty$.

$$\Delta(s) = \det J(s) \prod_{i=1}^{n-1} P'(\tau_i^s) \prod_{i=2}^{n-1} (P'(t_i^s - h_i) - P'(t_i^s + h_i)), \tag{3.2.2}$$

where

$$\det J(s) = \frac{R_s^n}{(r-1)!^n \prod_{i=1}^n 2h_i} \int_{t_1^s - 1}^{t_1^s + h_1} \cdots \int_{t_n^s - h_n}^{t_n^s + h_n} \det T(s)\, dz_1 \cdots dz_n$$

and $T(s)$ is the matrix

$$\begin{pmatrix} P(z_1) & u_0(z_1) & \cdots & u_{r-1}(z_1) & \varphi(z_1, \xi_1^s) & \cdots & (-1)^{n-r} \varphi(z_1, \xi_{n-r-1}^s) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ P(z_n) & u_0(z_0) & \cdots & u_{r-1}(z_n) & \varphi(z_n, \xi_1^s) & \cdots & (-1)^{n-r} \varphi(z_n, \xi_{n-r-1}^s) \end{pmatrix}.$$

Here $u_i(t) = t^i$ and $\varphi(z, \xi) = (z - \xi)_+^{r-1}$. Expanding $\det T(s)$ along the first column we get

$$\begin{aligned} \det T(s) = {} & \frac{R_s^n}{(r-1)!^n \prod_{i=1}^n 2h_i} \left(\sum_{i=1}^n (-1)^{i-1} \int_{t_i^s - h_i}^{t_i^s + h_i} P(z_i)\, dz_i \right) \\ & \times \int_{t_1^s - h_1}^{t_1^s + h_1} \cdots \int_{t_{i-1}^s - h_{i-1}}^{t_{i-1}^s + h_{i-1}} \int_{t_{i+1}^s - h_{i+1}}^{t_{i+1}^s + h_{i+1}} \cdots \int_{t_n^s - h_n}^{t_n^s + h_n} \\ & \times \det J_i(s)\, dz_1 \cdots dz_{i-1}\, dz_{i+1} \cdots dz_n, \end{aligned} \tag{3.2.3}$$

where $J_i(s)$ is the submatrix of

$$\begin{pmatrix} u_0(z_1) & \cdots & u_{r-1}(z_1) & \varphi(z_1, \xi_1^s) & \cdots & (-1)^{n-r} \varphi(z_1, \xi_{n-r-1}^s) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_0(z_n) & \cdots & u_{r-1}(z_n) & \varphi(z_n, \xi_1^s) & \cdots & (-1)^{n-r} \varphi(z_n, \xi_{n-r-1}^s) \end{pmatrix}$$

with the ith row eliminated. We have from the total positivity of the truncated power kernel (see [8]) and the fact in (2.21) that the intervals $(t_i^0 - h_i, t_i^0 + h_i)$ are disjoint that

$$\mu \det J_i(0) \geqslant 0, \tag{3.2.4}$$

for $\mu \in \{-1, 1\}$ fixed and $i = 1, ..., n$. Using (2.21) we see that

$$(-1)^{i-1} \int_{t_i^0 - h_i}^{t_i^0 + h_i} P_0(z_i)\, dz_i = \left| \int_{t_i^0 - 1}^{t_i^0 + 1_i} P_0(z_i)\, dz_i \right|, \qquad i = 1, ..., n. \tag{3.2.5}$$

We will show that $\det J_i(0) \neq 0$ for some $z_j \in (t_j^0 - h_j, t_j^0 + h_j)$, $j = 1, ..., n$, $j \neq i$, $i = 1, ..., n$. Let $i \in \{1, ..., n\}$ be fixed. Set $y_j = z_j$, $j = 1, ..., i-1$ and $y_j = z_{j+1}$, $j = i, ..., n-1$. Then $\det J_i(0) \neq 0$ if $y_j < \xi_j^0 < y_{j+r}$, $j = 1, ..., n-r-1$; i.e., the interlacing conditions hold. By the interlacing conditions between the zeros and the knots of P_0' we have

$$x_{i+1}^0 < \xi_i^0 < x_{i+r}^0, \qquad i = 1, ..., n-r-1, \tag{3.2.6}$$

where x_i^0, $i = 2, ..., n-1$, are the zeros of P_0'. By (2.21) $x_i^0 \in (t_i^0 - h_i, t_i^0 + h_i)$, $i = 2, ..., n-1$. Denote $x_1 = h_1$, $x_i = x_i^0$, $i = 2, ..., n-1$, $x_n = 1 - h_n$. Set $y_j = x_j$ if $1 \leqslant j \leqslant i-1$, and $y_j = x_{j+1}$ if $i \leqslant j \leqslant n-1$. Then from (3.2.6) we have

$$y_j < \xi_j^0 < y_{j+r}, \qquad j = 1, ..., n-r-1. \tag{3.2.7}$$

From (3.2.2), (3.2.3), (3.2.4), (3.2.5), and (3.2.7) we obtain

$$\Delta(0) \neq 0.$$

Hence, by the Implicit Function Theorem, there exists a unique system of continuous functions

$$a_0^s, ..., a_{r-1}^s, t_2^s, ..., t_{n-1}^s, \tau_1^s, ..., \tau_{n-1}^s, \xi_1^s, ..., \xi_{n-r-1}^s, R_s,$$

defined in a neighborhood of 0, which satisfy (3.2.1). Thus, we have found a solution $P(s, x)$ of (3.2.1) for small s. In order to prove that $\Delta(s) \neq 0$ for each $s \in [0, 1]$ we need to show that the intervals $(t_i(s) - h_i, t_i(s) + h_i)$, $i = 1, ..., n$, are disjoint; i.e.,

$$(t_i^s - h_i, t_i^s + h_i) \subset (\tau_{i-1}^s, \tau_i^s), \qquad i = 1, ..., n, \quad \tau_0^s = 0, \quad \tau_n^s = 1. \tag{3.2.8}$$

Define $P^*(s, x) = R_s P(s, x)$, $R_s = \|P^{*(r)}\|_\infty$. Analogous to (2.14) and (2.18) for the perfect spline P^* we have

$$R_s \leqslant \|P^*\|_\infty (n-1)^r\, 2^{2r-1}(r+2)\, r!, \tag{3.2.9}$$

$$\|P^{*\prime}\|_\infty \leqslant 2M\, \|P^*\|_\infty^{(r-1)/r}\, \|P^{*(r)}\|_\infty^{1/r}. \tag{3.2.10}$$

By (3.2.9), (3.2.10), it follows that

$$\|P^{*\prime}\|_\infty \leqslant 2M\, \|P^*\|_\infty\, c(r)(n-1). \tag{3.2.11}$$

From (3.2.1) and the Mean Value Theorem it follows that there exist points $y_i^s \in (t_i^s - h_i, t_i^s + h_i)$ such that $P^*(y_i^s) = (-1)^{i-1} e_i(s)$, $i = 1, ..., n$. Let x_i^s, $i = 2, ..., n-1$, be the extremal points of P^* and $x_1^s = 0$, $x_n^s = 1$. By the conditions $P^*(t_i^s - h_i) = P^*(t_i^s + h_i)$, $i = 2, ..., n-1$, in (3.2.1) it follows that $x_i^s \in (t_i^s - h_i, t_i^s + h_i)$, $i = 2, ..., n-1$. Let $\|P^*\|_\infty = |P(x_{i_0}^s)|$. Then

$$P^*(x_{i_0}^s) = P^*(y_{i_0}^s) + (x_{i_0}^s - y_{i_0}^s)\, P^{*\prime}(\eta),$$

where $\eta \in (y_{i_0}^s, x_{i_0}^s) \subset (t_{i_0}^s - h_{i_0}, t_{i_0}^s + h_{i_0})$. Therefore

$$\|P^*\|_\infty \leqslant e_{i_0}(s) + 2h_{i_0}\, \|P^{*\prime}\|_\infty. \tag{3.2.12}$$

By (3.2.11), (3.2.12), and the choice of h_{i_0} in (1.4) we obtain

$$\|P^*\|_\infty \leqslant \frac{4 \max\{e_i(s)\}_{i=1}^n \max\{e_i\}_{i=1}^n}{4 \max\{e_i\}_{i=1}^n - \min\{e_i\}_{i=1}^n} \leqslant (4/3) \max\{e_i(s)\}_{i=1}^n. \tag{3.2.13}$$

Applying (3.2.13) in (3.2.11), we get

$$\|P^{*\prime}\|_\infty \leqslant M(n-1)\, c(r)(8/3) \max\{e_i(s)\}_{i=1}^n. \tag{3.2.14}$$

We have

$$\min\{e_i(s)\}_{i=1}^n < |P^*(x_i^s)| \leqslant (\tau_i^s - x_i^s)\, \|P^{*\prime}\|_\infty, \qquad i = 1, ..., n-1,$$

$$\min\{e_i(s)\}_{i=1}^n < |P^*(x_i^s)| \leqslant (x_i^s - \tau_{i-1}^s)\, \|P^{*\prime}\|_\infty, \qquad i = 2, ..., n.$$

Using (3.2.14) in the last inequalities we obtain

$$\begin{aligned} \tau_i^s - x_i^s &> \frac{3 \min\{e_i(s)\}_{i=1}^s}{8 \max\{e_i(s)\}_{i=1}^n M(n-1)\, c(r)}, \qquad i = 1, ..., n-1, \\ x_i^s - \tau_{i-1}^s &= \frac{3 \min\{e_i(s)\}_{i=1}^n}{8 \max\{e_i(s)\}_{i=1}^n M(n-1)\, c(r)}, \qquad i = 2, ..., n. \end{aligned} \tag{3.2.15}$$

From (2.22), (2.23) and the definition of $e_i(s)$ we see that

$$\frac{3\min\{e_i(s)\}_{i=1}^n}{8\max\{e_i(s)\}_{i=1}^n} > \frac{9\min\{e_i\}_{i=1}^n}{32\max\{e_i\}_{i=1}^n} > \frac{\min\{e_i\}_{i=1}^n}{8\max\{e_i\}_{i=1}^n}. \tag{3.2.16}$$

Then by (3.2.15), (3.2.16), and (1.4), we get

$$\tau_i^s - x_{i-1}^s > 2h_i, \qquad i = 1, ..., n-1,$$
$$x_i^s - \tau_{i-1}^s > 2h_i, \qquad i = 2, ..., n.$$

Therefore

$$\tau_i^s - t_i^s - h_i > \tau_i^s - x_i^s - 2H_i > 0, \qquad i = 1, ..., n-1,$$
$$t_i^s - h_i - \tau_{i-1}^s > x_i^s - \tau_{i-1}^s - 2h_i > 0, \qquad i = 2, ..., n,$$

which means that (3.2.8) holds. Using the same arguments as in the proof that $\Delta(0) \neq 0$ we can show that $\Delta(s) \neq 0$ for each $s \in [0, 1]$. Now we can continue the proof as in Theorem 3.1 in Bojanov and Daren [4] and get the unique $P(x) = P(1; x)$, $\{t_i^* = t_i^1\}_{i=1}^n$ and $R(e_1, ..., e_n) = R_1$, satisfying the system (3.2.1). The details are omitted; see Bojanov and Daren [4].

It remains to show that $r(e_1, ..., e_n)$ is a strictly increasing function of e_i, $i = 1, ..., n$. Since $\Delta(1) \neq 0$, the Implicit Function Theorem implies that $r(e_1, ..., e_n)$ is a differentiable function of e_k. Moreover we have

$$\frac{\partial R}{\partial e_k} = \frac{\Delta_k}{\Delta(1)},$$

where Δ_k is obtained from $\Delta(1)$, replacing the column corresponding to R by the derivative of the right-hand side of (3.2.1), with respect to e_k. Denote by $\int_{I_k}$ the $n-1$ multiple integral over the intervals $[t_1 - h_1, t_1 + h_1]$, ..., $[t_{k-1} - h_{k-1}, t_{k-1} + h_{k-1}]$, $[t_{k+1} - h_{k+1}, t_{k+1} + h_{k-1}]$, ..., $[t_n - h_n, t_n + h_n]$. Using the same arguments as in the evaluation of $\Delta(0)$, we find

$$\frac{\partial R}{\partial e_k} = \frac{|\int_{I_k} \det J(1)\, dz_1 \cdots dz_{k-1}\, dz_{k+1} \cdots dz_n|}{\sum_{i=1}^n 2h_i e_i\, |\int_{I_i} \det J_i(1)\, dz_1 \cdots dz_{i-1}\, dz_{i+1} \cdots dz_n|},$$

which is positive for $k = 1, ..., n$. This completes the proof of Theorem 3.

THEOREM 4. *There exists a unique perfect spline P^* of degree r with $n-r-1$ knots and a unique set of points $\mathbf{t}^* \in \Xi_n^{\mathbf{h}}$ such that $P^* \in W_\infty^r(\mathbf{t}^*; \mathbf{h}; \mathbf{e})$. P^* is uniquely characterized by the conditions*

$$P^*(t_i^* - h_i) = P^*(t_i^* + h_i), \qquad i = 2, ..., n-1,$$

and $t_1^* = h_1$, $t_n^* = 1 - h_n$. *This perfect spline is the unique solution to* (1.5) *for* $p = \infty$.

Proof. It follows from Theorem 3 that there is unique perfect spline P^*, points $\mathbf{t}^* \in \Xi_n^{\mathbf{h}}$ and a constant R such that

$$P^* \in W_\infty^r(\mathbf{t}^*; \mathbf{h}; \mathbf{e}),$$

$$P^*(t_i^* - h_i) = P^*(t_i^* + h_i), \qquad i = 2, ..., n-1,$$

$$\|P^{*(r)}\|_\infty = R(e_1, ..., e_n).$$

Now suppose that there exists a perfect spline P of degree r with $n - r - 1$ knots such that $P \not\equiv P^*$, $P \in W_\infty^r(\mathbf{t}^*; \mathbf{h}; \mathbf{e})$ for some point $\mathbf{t} \in \Xi_n^h$, $\mathbf{t} \neq \mathbf{t}^*$, and

$$\|P^{(r)}\|_\infty \leqslant \|P^{*(r)}\|_\infty. \tag{3.2.17}$$

$P(x)$ has exactly $n-1$ zeros: $\tau_1, ..., \tau_{n-1}$. Let $\tau_0 = 0$, $\tau_n = 1$. If $\tau_i - \tau_{i-1} \geqslant 2h_i$ there exists a point $\eta_i \in (\tau_{i-1}, \tau_i)$ such that

$$P(\eta_i - h_i) = P(\eta_i + h_i), \qquad \text{if} \quad i \in \{2, ..., n-1\},$$

$$\eta_1 = h_1, \qquad \eta_n = 1 - h_n, \qquad \text{if} \quad i = 1 \quad \text{or} \quad i = n,$$

$$\frac{1}{2h_i} \int_{\eta_i - h_i}^{\eta_i + h_i} P(x)\, dx = (-1)^{i-1} d_i.$$

The point η_i is the extremal point of the function $F(x) = \int_{x-h_i}^{x+h_i} P(t)\, dt$ in the interval (τ_{i-1}, τ_i) if $i \in \{2, ..., n-1\}$. If $i = 1$ or $i = n$ we have $F(x)$ is strictly monotone in the intervals $[h_1, \tau_1]$, $[\tau_{n-1}, 1 - h_n]$ and $|F(x)|$ attains maximal values for $x = \eta_1$ and $x = \eta_n$, respectively. Therefore $d_i \geqslant e_i$. If $\tau_i - \tau_{i_1} < 2h_i$ we have

$$e_i < \left| \frac{1}{2h_i} \int_{x_i}^{y_i} P(t)\, dt \right| = \frac{y_i - x_i}{2h_i} |P(z_i)|,$$

where $[x_i, y_i] \subseteq [\tau_{i-1}, \tau_i]$ and $z_i \in (x_i, y_i)$. Denote by $2h_i' = y_i - x_i$, $h_i' < h_i$. Then

$$\frac{h_i}{h_i'} e_i < |P(z_i)|. \tag{3.2.18}$$

Therefore there exists a point $\eta_i \in (\tau_{i-1}, \tau_i)$ such that

$$\begin{aligned} &P(\eta_i - h'_i) = P(\eta_i + h'_i), && \text{if} \quad i \in \{2, ..., n-1\}, \\ &\eta_1 = h'_1, \quad \eta_n = 1 - h'_n, && \text{if} \quad i = 1 \quad \text{or} \quad i = n, \\ &\frac{1}{2h'_i} \int_{\eta_i - h'_i}^{\eta_i + h'_i} P(t)\, dt = (-1)^{i-1}\, d_i. \end{aligned}$$

From (3.2.18) $d_i \geqslant |p(z_i)| > (h_i/h'_i)\, e_i$. The perfect spline $P(x)$ satisfies the following conditions

$$\begin{aligned} \frac{1}{2h'_i} \int_{\eta_i - h'_i}^{\eta_i + h'_i} P(x)\, dx &= (-1)^{i-1}\, d_i, && i = 1, ..., n, \\ P(\eta_i - h'_i) &= P(\eta_i + h'_i), && i = 2, ..., n-1, \\ \|P^{(r)}\| &= R(d_1, ..., d_n), \end{aligned} \tag{3.2.19}$$

where $d_i \geqslant e_i$, $h'_i = h_i$ if $\tau_i - \tau_{i-1} \geqslant 2h_i$ and $d_i > (h_i/h'_i)\, e_i$, $h'_i < h_i$ if $\tau_i - \tau_{i-1} < 2h_i$.

Consider $P^*(x)$ in the interval $[t_i^* - h_i, t_i^* + h_i]$. From Theorem 3 we have that $[t_i^* - h_i, t_i^* + h_i] \subset [\tau_{i-1}^*, \tau_i^*]$, where τ_i^*, $i = 1, ..., n-1$, are the zeros of P^* and $\tau_0^* = 0$, $\tau_n^* = 1$. Therefore there exist points $r_i \in (t_i^* - h_i, t_i^* + h_i)$ such that

$$\begin{aligned} c_i &= \frac{1}{2h'_i} \left| \int_{r_i - h'_i}^{r_i + h'_i} P^*(x)\, dx \right| \\ &\leqslant \frac{1}{2h'_i} \left| \int_{t_i^* - h_i}^{t_i^* + h_i} P^*(x)\, dx \right| = \frac{h_i}{h'_i}\, e_i, && i = 1, ..., n, \\ P^*(r_i - h'_i) &= P^*(r_i + h'_i), && i = 2, ..., n-1, \\ \|P^{*(r)}\| &= R(c_1, ..., c_n), \end{aligned} \tag{3.2.20}$$

where $r_i = t_i^*$ if $h_i = h'_i$. We have $d_i > c_i$ at least for one i since $P \not\equiv P^*$. From (3.2.19), (3.2.20) and the strong monotonicity of R given by Theorem 3 we obtain

$$\|P^{*(r)}\|_\infty < \|P^{(r)}\|_\infty,$$

which contradicts (3.2.17).

It remain to prove that P^* is the unique solution to problem (1.5) for $p = \infty$. We will apply a similar approach to that in [13]. Assume that $f \in W_\infty^r(\mathbf{t}; \mathbf{h}; \mathbf{e})$ for some point $\mathbf{t} \in \Xi_n^{\mathbf{h}}$. From Theorem 1 there exists a perfect spline P of degree r with $n-r-1$ knots for which $(1/2h_i)$ $\int_{t-i-h_i}^{t_i+h_i} P(x)\,dx = (1/2h_i)\int_{t_i-h_i}^{t_i+h_i} f(x)\,dx$, $i=1, ..., n$, and $\|P^{(r)}\|_\infty \leqslant \|f^{(r)}\|_\infty$. If $\mathbf{t} \neq \mathbf{t}^*$, then $\|P^{*(r)}\|_\infty < \|P^{(r)}\|_\infty$. Thus if $f \in W^r_\infty[0, 1]$ is a solution of (1.5), it is necessary that

$$\|f^{(r)}\|_\infty = \|P^{*(r)}\|_\infty \tag{3.2.21}$$

and $f \in W^r_\infty(\mathbf{t}^*; \mathbf{h}; \mathbf{e})$.

Now we will prove $f(t_i^* - h_i) = f(t_i^* + h_i)$, $i = 2, ..., n-1$, for any f as above. Assume that $f(t_j^* - h_j) \neq f(t_j^* + h_j)$ for some $j \in \{2, ..., n-1\}$. From the Mean Value Theorem we have $f(x_j) = (-1)^{j-1} e_j$ for some point $x_j \in (t_j^* - h_j,\ t_j^* + h_j)$. By the character of the data there exist points $\tau_{j-1} < x_j < \tau_j$ such that $f(\tau_{j-1}) = 0$ and $f(\tau_j) = 0$. Let τ_{j-1}, τ_j be the nearest zeros of f to the point x_j. There is a point $\eta_j \in (\tau_{j-1}, \tau_j)$ and a number $h'_j \in (0, h_j]$ such that

$$f(\eta_j - h'_j) = f(\eta_j + h'_j),$$

$$\frac{1}{2h'_i} \int_{\eta_j - h'_j}^{\eta_j + h'_j} f(x)\,dx = (-1)^{j-1} d_j.$$

Analogously to (3.2.18) we see that $d_j > (h_j/h'_j)\,e_j$. Then using similar arguments to those in (3.2.19) and (3.2.20) we can show that $\|P^{*(r)}\|_\infty < \|f^{(r)}\|_\infty$, which contradicts the minimality property of f. Thus $f(t_i^* - h_i) = f(t_i^* + h_i)$, $i = 2, ..., n-1$.

Assume that $f \not\equiv P^*$. Let σ be small. For $0 < \sigma < 1$

$$|P^{*(r)}(x)| = \|P^{*(r)}\|_\infty > (1-\sigma)\,|f^{(r)}(x)|$$

for each $x \in [0, 1]$. Thus $P^* - (1-\sigma) f$ cannot be a constant on any subinterval of $[0, 1]$. For the appropriate σ the derivative $(P^* - (1-\sigma) f)'$ has a sign change in each $(t_i^* - h_i, t_i^* + h_i)$ because the function takes the same value at the endpoints $t_i^* - h_i$, $t_i^* + h_i$ and it is not identically a constant on this interval. If $P^* - f$ is not identically zero in some $(t_j^* - h_j, t_{j+1}^* + h_{j+1})$, $j \in \{2, ..., n-1\}$, then since

$$\int_{t_j^* - h_j}^{t_j^* + h_j} [P^*(x) - f(x)]\,dx = 0,$$

$$(P^* - f)(t_j^* - h_j) = (P^* - f)(t_j^* + h_j),$$

$$\int_{t_{j+1}^* - h_{j+1}}^{t_{j+1}^* + h_{j+1}} [P^*(x) - f(x)]\,dx = 0,$$

and

$$(P^* - f)(t_{j+1}^* - h_{j+1}) = (P^* - f)(t_{j+1}^* + h_{j+1}),$$

$P-f$ has at least two zeros in $(t_j^*-h_j, t_j^*+h_j]$ and at least two zeros in $[t_{j+1}^*-h_{j+1}, t_{j+1}^*+h_{j+1})$. Therefore $(P^*-f)'$ and thus $(P^*-(1-\sigma)f)'$ has at least three sign changes in $(t_j^*-h_j, t_{j+1}^*+h_{j+1})$. Together we now have at least $n-1$ sign changes of $(P^*-(1-\sigma)f)'$. Then by Rolle's Theorem $(P^*-(1-\sigma)f)^{(r)}$ has at least $n-r$ sign changes on $[0, 1]$, which is a contradiction because $(P^*-(1-\sigma)f)^{(r)}$ has the same number of sign changes as $P^{*(r)}$; i.e., exactly $n-r-1$. So now assume P^*-f is identically zero on $[t_2^*-h_2, t_{n-1}^*+h_{n-1}]$. If $(P^*-f)'$ has a sign change in $[t_1^*-h_1, t_2^*-h_2]$ we proceed as above. If $(P^*-f)'$ does not change sign in $[t_1^*-h_1, t_2^*-h_2]$, then P^*-f is monotone on this interval. Since P^*-f vanishes at t_2-h_2 while the integral over $[t_1^*-h_1, t_1^*+h_1]$ is zero, this implies that P^*-f is identically zero on $[t_1^*-h_1, t_2^*-h_2]$. We use the same argument on $[t_{n-1}^*+h_{n-1}, t_n^*+h_n]$. This proves the uniqueness.

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